AIAA 81-4274

Finite Elements for Initial Value Problems in Dynamics

T. E. Simkins*
U.S. Army Armament Research and Development Command, Watervliet, N.Y.

The work of C. D. Bailey amply demonstrates that a variational principle is not a necessary prerequisite for the formulation of variational approximations to initial value problems in dynamics. While Bailey successfully applies global power series approximations to Hamilton's Law of Varying Action, the work herein shows that a straightforward extension to finite element formulations fails to produce a convergent sequence of solutions. The source of the difficulties and their elimination are discussed in some detail, and a workable formulation for initial value problems is obtained. The paper concludes with a few elementary examples showing the utility of finite elements in the time domain.

I. Introduction

According to Finlayson and Scriven,[1] it is not variational notation or even the concept of a varied path which is the key criterion of a true variational "principle," but rather the existence of a functional which, when varied and set to zero, generates the governing equations and constraints for a given class of problems. In this sense, certain fundamental principles of mechanics, such as d'Alembert's Principle, do not truly qualify as variational principles. That is to say, these mechanical principles or "laws" cannot be posed as central problems of the calculus of variations. On the other hand, there are others, such as Hamilton's principle, which do qualify as true variational principles. Yet it is d'Alembert's Principle which forms a basis for all analytical mechanics,[2] and it follows, therefore, that the vanishing of the first variation of some functional is not a necessary condition for the scalar formulation of any mechanics problem—however elegant or convenient this may be. Whether a true variational principle or a more fundamental variational statement is used to obtain a numerical solution to a dynamics problem, an important argument is that well established laws such as d'Alembert's Principle or true principles such as Hamilton's are physically based and avoid the arbitrariness inherent in general weighted residual methods and contrived variational principles. Moreover, only those variational principles which are also maximum or minimum principles appear to offer any special advantage for obtaining approximate solutions—mainly through their ability to provide bounds on the variational integral. Even then the system treated must be positive-definite and the upper and lower bounds are often too far apart to be of practical value. In brief, there seems to be little point in contriving a variational principle in preference to a variational law of mechanics, despite the more primitive status of the latter. Indeed the many solutions to initial value dynamics problems achieved by Bailey[3] by applying the Ritz method to Hamilton's "law of varying action" demonstrate the usefulness of variational formulations not qualifying as "principles." Thus motivated, the work herein explains the numerical difficulties encountered in attempting to generalize Bailey's formulations according to the method of finite elements.

Zienkiewicz[4] has expressed serious reservations about the use of finite elements in the time domain. Indeed, when the functions involved are sufficiently smooth, the number of time steps required to integrate a set of ordinary differential equations may not be great, and it may require roughly as many finite elements to produce a solution of comparable accuracy. In view of the increased storage required, the use of time-finite elements to solve such systems is questionable. There are many other cases, however, in which conventional algorithms for step-by-step integration may call for a very large number of time steps. This is especially true when dealing with the (hyperbolic) equations of structural dynamics should the excitation and/or material properties change rapidly in time. A physically based variational method, with its inherent stability and physical origin, may lower the computational effort considerably.

The many solutions achieved by Bailey were generated by the Ritz method[5] using a power series approximation in which globally defined polynomials are the basis functions. Ultimately, the length of interval over which solutions may be generated, as well as the detail to be provided in any subinterval, will be limited by the degree of polynomial used as a basis. The pitfalls of using higher powered polynomials are well documented[6] and partially account for the use of locally (piecewise) defined basis functions (finite elements) to solve problems in many branches of mathematical physics. The extraordinary accuracy and simplicity of procedure attained by Bailey, however, are, not to be understated.

Apart from avoiding the problems which can arise when higher powered polynomials are employed as basis functions, finite element formulations have other advantages when used to solve problems in continuum mechanics. Even though the principal motivation for their use has been the need to handle complicated boundary shapes (nonexistent in the time domain), time-finite elements are also well suited to handle sudden changes in load functions, extending the interval of solution indefinitely without restart, and providing great detail to the solution in any subinterval. Two examples which exploit the advantages afforded by the finite element discretization of time are given in Sec. V.

Since 1977, several investigations dealing with the use of finite elements to modify or replace conventional integration methods have been published. Hughes and Liu[7] and Belytschko and Mullen[8] are notable examples. One also notes the work of Serbin et al., who have recently begun a computational and theoretical study of finite element methods for hyperbolic equations.[9] Thus, despite the reservations expressed by Zienkiewicz, the extension of the finite element method to the solution of transient field problems is well motivated, and was first reported by Argyris and Sharpf,[10] later by Fried,[11] and most recently by Baruch and Riff.[12,13] All of these works attempt to use Hamilton's principle as a starting point for the finite element formulation of initial value problems. As will be pointed out in the following section, this cannot be accomplished without some logical

Received Sept. 11, 1980; revision received March 23, 1981.

*Research Mechanical Engineer, Dept. of Applied Mathematics and Mechanics, Large Caliber Weapon Systems Laboratory, Benet Weapons Laboratory.

inconsistency when bringing the initial data into the formulation. In the sequel it will be shown that the use of Hamilton's "law," rather than Hamilton's "principle," makes possible the logical incorporation of the initial conditions into the variational formulation.

II. Hamilton's Principle—A Constrained Variational Principle

The following equation is known as the generalized principle of d'Alembert[14]:

$$\sum_{i=1}^{N} (F_i - \dot{P}_i) \cdot \delta r_i = 0; \quad (\dot{\ }) = \partial/\partial t \tag{1}$$

This equation applies to any system of N particles, the ith particle having a position r_i, a momentum P_i, and subject to a resultant applied force F_i.

Under the assumption that the virtual work of the applied forces is derivable from a scalar V, a time integration of Eq. (1) leads to Hamilton's law of varying action[15,16]:

$$\delta \int_{t_1}^{t_2} (T-V)\,\mathrm{d}t - \sum_{i=1}^{N} m_i \dot{r}_i \cdot \delta r_i \Big]_{t_1}^{t_2} = 0 \tag{2a}$$

T is the kinetic energy of the system

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i \cdot \dot{r}_i$$

and V is the potential energy of the forces impressed on the N particles. The existence of V makes little difference as far as numerical calculations are concerned. In the event V does not exist, Eq. (2a) can be written:

$$\int_{t_1}^{t_2} (\delta T + \delta \overline{W})\,\mathrm{d}t - \sum_{i=1}^{N} m_i \dot{r}_i \cdot \delta r_i \Big]_{t_1}^{t_2} = 0 \tag{2b}$$

The bar signifies that in general the virtual work of the applied forces cannot be derived from any scalar function of the generalized coordinates. Either of Eqs. (2) can be used as a basis for a Ritz approximation to a dynamics problem.

If the r_i are *constrained* to take on specified values at t_1 and t_2, *then* $\delta r_i(t_1)$ and $\delta r_i(t_2)$ vanish in Eq. (2a) and the result is Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T-V)\,\mathrm{d}t = 0 \tag{3}$$

Since the vanishing of the displacement variations at the end points is not the only means by which the partial sum in Eq. (2a) may vanish, Eq. (3) may not always represent Hamilton's principle in the strict sense. Should Eq. (3) be used as a basis for the numerical solution of a dynamics problem without imposing the constraint that all of the δr_i vanish at t_1 or t_2, zero momentum conditions will prevail instead as natural boundary conditions on those displacements whose variations are free. This aspect of variational principles is covered very clearly in many references (cf. Ref. 17). An observation to be made here is that Eq. (3) corresponds to a system of boundary value problems—not initial value problems—since the partial sum can only vanish through boundary (end point) constraints either natural or imposed. Thus Eq. (3) cannot, with complete logic, be used to formulate any system of initial value problems of dynamics. The introduction of initial data has in fact always been the obstacle preventing the use of Hamilton's principle for the variational formulation of initial value problems.[18,19]

Since Hamilton's principle is a valid physical statement of mechanics only when the boundary constraints are such that the partial sum vanishes, it is really a "constrained variational principle," as opposed to Eqs. (2), which are unconstrained variational laws of mechanics, suitable for the application of arbitrary constraint conditions.

III. Global and Piecewise Ritz Approximations

Equations (2) and (3) differ only in the presence or absence of boundary terms. For the case of a single particle ($N=1$) having only one degree-of-freedom $u(t)$, the Ritz procedure, when applied to either of Eqs. (2), leads to a scalar relation of the form:

$$\delta U^T[(K-B)U-F]=0 \tag{4}$$

whereas for Eq. (3):

$$\delta U^T[KU-F]=0 \tag{5}$$

Equations (4) and (5) are assumed to derive from applying the Ritz procedure whereby the displacement function $u(t)$ is approximated as:

$$u(t) \doteq a^T(t)\,U \tag{6}$$

Relation (6) applies to the entire interval of solution when globally defined basis functions are used or to a particular subinterval thereof when piecewise functions (finite elements) are employed. When a global power series approximation is used, U is a vector of generalized coordinates, the first two of which are identifiable as $u(t_1)$ and $\dot{u}(t_1)$. The "shape function" $a(t)$ in this case is simply:

$$a^T(t) = [1, t, t^2, \ldots, t^n], \quad t_1 \le t \le t_2 \tag{7}$$

If piecewise cubic Hermite polynomials are used instead, the components of U are local values of u and $\dot{u}$ defined at the endpoints of a particular subinterval, and

$$a^T(t) = [2\tau^3 - 3\tau^2 + 1,\ h(\tau^3 - 2\tau^2 + \tau),$$
$$3\tau^2 - 2\tau^3,\ h(\tau^3 - \tau^2)] \tag{8}$$

where $\tau = t/h$, h being the length of the particular subinterval. Referring first to Eq. (5), it is noted that K tends to be singular of degeneracy one. For certain simple problems K may compute to be exactly singular. In general, however, K will only become singular in the limit as the number of basis functions employed in the Ritz approximation becomes infinite. The degeneracy of K represents the possibility that neither $u(t_1)$ or $u(t_2)$ has been specified. That is, if neither $\delta u(t_1)$ or $\delta u(t_2)$ vanishes, then $m\dot{u}$ must vanish at both endpoints as natural boundary conditions. Under these conditions $u(t)$ may only be determined to within an arbitrary constant. Thus in Eq. (5) K may only be reduced to a nonsingular matrix by specifying values for $u(t_1)$ and/or $u(t_2)$ so that the variations of one or both of these quantities vanish. The essence of the discussion which follows is not changed if, in the sequel, it is assumed that $u(t_1)$ has been specified. This is known as a "geometric" or "imposed" constraint. Because $\delta U_1 \equiv \delta u(t_1) = 0$ multiplies the first row of K in Eq. (5), this row is effectively removed from the formulation. Since the remaining variations are arbitrary, the final set of equations to be solved is then:

$$\sum_{j=2}^{n} K_{ij} U_j = F_i - K_{i1} U_1, \quad i = 2, 3, \ldots n \tag{9}$$

where $U_1 = u(t_1)$ is the specified value and $n \times n$ is the dimension of K. Whether these equations derive from a global power series approximation or from one based on finite elements, one may readily verify that as n is increased their solutions do indeed converge to the exact solution of the corresponding two point time-boundary value problem. Should one wish a solution to an initial value problem, however, Eq. (4) must be used instead of Eq. (5). In this case, specifying values for $u(t_1)$ and $\dot{u}(t_1)$ cause δU_1 and δU_2 to vanish, thereby deleting the first two equations of this set. The resulting system of equations to be solved is thus:

$$\sum_{j=3}^{n} (K_{ij}-B_{ij})\,U_j = F_i - (K_{i1}-B_{i1})\,U_1 - (K_{i2}-B_{i2})\,U_2, \quad i=3,4,\ldots,n \tag{10}$$

In all cases attempted to date, solutions to Eqs. (10) have been observed to converge to the exact solution if these equations are derived using a global power series approximation but not if they are formulated by finite elements. An example of this anomaly will be given in the next section. As the only difference between Eqs. (4) and (5) is a subtraction of $\boldsymbol{B}$ in the former, and inasmuch as convergence is achieved when Eq. (4) derives from a power series approximation, one suspects that it is the finite element representation of the matrix $\boldsymbol{B}$ which is somehow at fault. It is therefore of interest to know in more detail just how the subtraction of $\boldsymbol{B}$ is *supposed* to affect the coefficient matrix of the system.

In contrast to the matrix $\boldsymbol{K}$, the matrix $\boldsymbol{K}-\boldsymbol{B}$ must tend to be singular of degeneracy two—no constraints having been assumed a priori. Thus when $u(t_1)$ is specified and the first row of $\boldsymbol{K}-\boldsymbol{B}$ is deleted, the remaining equations still must possess one degeneracy in the limit as the number of basis functions becomes infinite. Thus the effect of subtracting $\boldsymbol{B}$ must be to free the natural boundary condition at t_2 [inherent in Eq. (5)] and to introduce a degeneracy. This remaining degeneracy can only be removed by specifying the value of $u(t)$ at a time other than t_1 or a value for $\dot{u}$, resulting in the deletion of another row of $\boldsymbol{K}-\boldsymbol{B}$.

IV. Anomalous Behavior of Finite Element Formulations

The degree to which the subtraction of the matrix $\boldsymbol{B}$ from $\boldsymbol{K}$ can both free the natural boundary condition at t_2 and introduce a degeneracy differs with the type of approximation employed. When global power series approximations are used, the $\boldsymbol{B}$ matrix is quite full and the subtraction affects many rows of $\boldsymbol{K}$. When locally defined Hermite polynomials are used, however, $\boldsymbol{B}$ is very sparse and in fact contains only two nonzero components. Moreover, one of these appears in the first row of $\boldsymbol{B}$ that is deleted when $u(t_1)$ is specified. In this case freeing the natural boundary condition and introducing a degeneracy depends on the subtraction from a single component of $\boldsymbol{K}$. Even though both effects may actually be produced in the limit as the number of elements becomes infinite, the degree to which they are approximated for any finite number of elements is evidently insufficient and the solutions do not converge to the correct result. This is exemplified in Fig. 1. The problem represented is that of a free oscillator of unit mass and stiffness, subject to the prescribed initial constraints of zero displacement and unit velocity. For this case, Eq. (2a) reads:

$$\int_0^{\pi} (\dot{u}\delta\dot{u} - u\delta u)\,\mathrm{d}t - \dot{u}\delta u\Big|_0^{\pi} = 0 \tag{11}$$

or simply,

$$\int_0^{\pi} (\ddot{u}+u)\,\delta u\,\mathrm{d}t = 0 \;\dagger \tag{12}$$

The finite element results of Fig. 1 were obtained using piecewise cubic Hermite polynomials. (Higher ordered Hermite polynomials yield similar results.) It is observed that the solutions tend to diminish from the exact solution, $\sin(t)$, as the number of elements is increased. Using only two finite elements, the finite element matrix formulation [Eq. (4)] for

†Note that Eq. (12) would also result from application of the Galerkin procedure, implying that the Galerkin method has some physical justification for problems in dynamics.

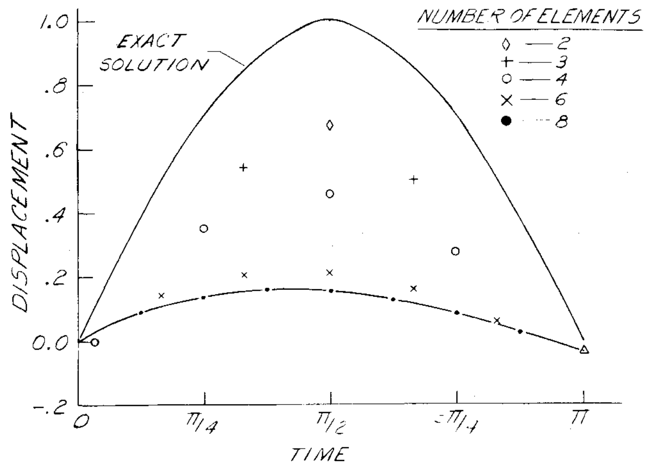

Fig. 1 Divergent finite element solutions to free oscillator problem.

this problem is as follows:

$$0=\delta U^T[\boldsymbol{K}-\boldsymbol{B}]U=[\delta U_1\;\delta U_2\;\delta U_3\;\delta U_4\;\delta U_5\;\delta U_6]$$

$$\cdot\left\{\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & 0 & 0\\ k_{21} & k_{22} & k_{23} & k_{24} & 0 & 0\\ k_{31} & k_{32} & k_{33}+k_{11} & k_{34}+k_{12} & k_{13} & k_{14}\\ k_{41} & k_{42} & k_{43}+k_{21} & k_{44}+k_{22} & k_{23} & k_{24}\\ 0 & 0 & k_{31} & k_{32} & k_{33} & k_{34}\\ 0 & 0 & k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\} \cdot \begin{bmatrix} U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6 \end{bmatrix} \tag{13}$$

Using expression (8), the element matrix $\boldsymbol{k}$ is calculated in terms of the element length h as:

$$\boldsymbol{k}=\int_0^h (\dot{a}\dot{a}^T - aa^T)\,\mathrm{d}t$$

$$=\begin{bmatrix} \frac{6}{5h}-\frac{13h}{35} & \frac{1}{10}-\frac{11h^2}{210} & -\frac{9h}{70}-\frac{6}{5h} & \frac{13h^2}{420}+\frac{1}{10}\\ & \frac{2h}{15}-\frac{h^3}{105} & -\frac{13h^2}{420}-\frac{1}{10} & \frac{h^3}{140}-\frac{h}{30}\\ -\text{SYMM.}- & & \frac{6}{5h}-\frac{13h}{35} & \frac{11h^2}{210}-\frac{1}{10}\\ & & & \frac{2h}{15}-\frac{h^3}{105} \end{bmatrix}$$

Since U_1 is specified the first row of $\boldsymbol{K}-\boldsymbol{B}$ is deleted. As the subtraction of $\boldsymbol{B}$ only affects one row of the reduced system, the only way in which a degeneracy can be introduced is for

Table 1 Solutions to free oscillator problem (displacement/velocity) $0 \le t \le \pi$

$6t/\pi$	One element	Two elements	Six elements	Exact solution
0	0.0[a]	0.0[a]	0.0[a]	0.0
	1.0[a]	1.0[a]	1.0[a]	1.0
1			0.49978005	0.5
			0.86602547	0.86602541
2			0.86564452	0.86602541
			0.50000025	0.5
3		0.97817298	0.99956036	1.0
		2.02985945E−4	4.4572957E−7	0.0
4			0.86564496	0.86602541
			−0.49999948	−0.5
5			0.499780823	0.5
			−0.86602502	0.86602541
6	0.0166090783	3.9845105E−4	8.9120273E−7	0.0
	−1.00079414	−1.00000946	−0.99999999	−1.0

[a] Imposed values.

the next to last row to join the space defined by the rows remaining. Thus rows 2-6 in Eqs. (13) ideally would become linearly dependent. This dependency among rows must be quite general, since specification of any other of the U_i must remove it.

One suspects that a simple subtraction of unity from K_{56} in Eqs. (13) may not do the best job of introducing a degeneracy or of freeing the natural boundary condition at $t_2 = \pi$. One can gain some idea of how "close" this subtraction brings the fifth row into the space of rows 2, 3, 4, and 6 by comparing it with its projection onto this space. Substituting $\pi/2$ for h, the fifth row of Eq. (13) calculates to be:

[0.0 0.0 −0.96590326 −0.17637194

0.180505097 −0.970755175]

whereas its projection is:

[7.8587145E−3 −8.5978979E−3 −0.974496335

−0.184380835 0.172642877 −0.96178340]

Further calculations show that if the interval of solution remains fixed and the number of finite elements is allowed to increase, closer agreement between the next to last row vector and its projection is observed, but this is not accompanied by a convergence of the solution vector toward the exact solution to the problem. While the exact reasons for this instability are not known, it is apparent that the rate at which the next to last row tends to become dependent is important. It stands to reason, therefore, that should one invoke the limit condition without actually proceeding to the limit, a convergent sequence may result and indeed this proves to be the case.

Asserting that the row vectors 2-6 are linearly dependent allows the fifth row (equation) of Eqs. (13) to be replaced by a linear combination of the others. For example, let

$$R_5 = \alpha_2 R_2 + \alpha_3 R_3 + \alpha_4 R_4 + \alpha_6 R_6 \tag{14}$$

where R_i denotes the ith row of $K - B$. After imposing the second initial constraint, $U_2 = 1$, Eqs. (13) can be written:

$$\delta U_3 R_3 \cdot U + \delta U_4 R_4 \cdot U + \delta U_5 (\alpha_2 R_2 + \alpha_3 R_3 + \alpha_4 R_4 + \alpha_6 R_6) \cdot U + \delta U_6 R_6 \cdot U = 0 \tag{15}$$

Since all variations in Eq. (15) are arbitrary, there results the following system of equations for solution:

$$0 = R_3 \cdot U = R_2 \cdot U = R_4 \cdot U = R_6 \cdot U \tag{16}$$

Thus the second equation (row), which was originally deleted through the specification of U_2, is brought back into the formulation in place of the fifth in a logical and consistent manner. Equations (16) are the same set as would result from following the procedure of Argyris and Scharpf.[20] These authors, however, started with Hamilton's principle, which requires that $\delta U_1 = \delta U_5 = 0$. This would delete the first and fifth equations from the set. Further specification of U_2 should then delete the second equation as well, overspecifying the problem. Argyris and Scharpf allow this equation to remain without justification. Moreover, no explanation is given as to why δU_5 should vanish, as U_5 is never specified in an initial value problem. All of these inconsistencies derive from the fact that Hamilton's principle corresponds only to boundary value problems—never to initial value problems.

In summary, the work of this section shows that Hamilton's law of varying action, unlike Hamilton's principle, is an unconstrained variational statement permitting the introduction of arbitrary constraints including data ordinarily given for initial value problems. When piecewise Hermite cubic polynomials are used as a basis for a finite element formulation, the singular state of the resulting coefficient matrix in the limit justifies retention of the second equation of the system, in preference to the next to last when typical initial values for displacement and velocity are specified. Following this procedure, convergent solutions are then obtained for the problem of the free oscillator considered in this section. These results are presented in Table 1 for formulations based on one, two, and six finite elements.

V. Applications

Example 1. Linear Oscillator Subjected to Discontinuous Forces

A linear oscillator of unit mass and stiffness is subjected to a force $f(t)$. Two cases are considered:

a) $f(t) = H(t - ½)$

b) $f(t) = \delta(t - ½)$

H and δ are the Heaviside and Dirac functions, respectively, and for either of these cases Eq. (2) reads:

$$\int_{t_1}^{t_2} \{\dot{u}\delta\dot{u} + (f(t) - u)\delta u\} \mathrm{d}t - \dot{u}\delta u \Big|_{t_1}^{t_2} = 0 \tag{17}$$

For case (a) four finite elements of equal length are used to approximate $u(t)$ over the solution interval (0,2). The element polynomial shape function is Hermite cubic and an element length of one half takes advantage of the specific

Table 2 Solution to $\ddot{u}+u=H(t-\frac{1}{2})$
$0\leq t\leq 2.0$

t	Computed Displacement	Computed Velocity	Exact Displacement	Exact Velocity
0.0	0.0[a]	1.0[a]	0.0	1.0
0.5	0.47932149	0.87708716	0.47942555	0.877582565
1.0	0.96370936	1.0199163	0.96388844	1.01972786
1.5	1.45700388	0.91238744	1.45719267	0.91220819
2.0	1.83836447	0.5805616	1.83856024	0.58134814

[a] Imposed values.

Table 3 Solution to $\ddot{u}+u=\delta(t-\frac{1}{2})$
$0\leq t\leq 1$

t	Computed displacement	Exact displacement
0.0	0.0[a]	0.0
0.1	0.1[a]	0.099833416
0.2	0.199001664	0.19866933
0.3	0.296016622	0.295520213
0.4	0.390076343	0.38941834
0.5	0.58007539	0.57925896
0.6	0.76428335	0.76331182
0.7	0.94086118	0.93973791
0.8	1.10804607	1.10677443
0.9	1.26416892	1.26275246
1.0	1.40767112	1.40611348

[a] Imposed values.

shape of the forcing function. Table 2 compares the calculated displacements and velocities with those computed from the exact solution.

In case (b) a discontinuity in velocity can be expected in the solution. As the use of cubic shape functions enforces continuity of velocity throughout, a better solution might be expected when linear shape functions are employed. Table 3 compares the exact solution on the interval (0,1) with that obtained using 10 such elements of equal length.

The two problems considered in this example demonstrate the manner in which the type of element (i.e., shape function) and its points of attachment (i.e., the nodes or grid points) may be varied to suit specified transient events.

Example 2. Response of a Beam to a Moving Mass

A concentrated mass is assumed to move at constant velocity v along the length of a uniform Euler beam, simply supported at each of its ends and having zero displacement and velocity at $t=0$. Under suitable definitions for k and m, the representative equations may be written[21]:

$$y^{iv}+k\ddot{y}+f(x,t)=0$$

$$y(0,t)=y''(0,t)=y(1,t)=y''(1,t)=y(x,0)=\dot{y}(x,0)=0 \quad (18)$$

The function $f(x,t)$ consists of a sum of inertial terms:

$$f(x,t)=m(\ddot{y}+2v\dot{y}'+g+v^2y'')\delta(x-vt) \quad (19)$$

where g denotes the gravitational constant and δ is the Dirac function. This problem is particularly interesting in that the conventional use of piecewise cubic shape functions, to discretize the space variable only, introduces forces which are discontinuous functions of time into the resulting ordinary differential equations. These discontinuities are associated with the beam curvature load term appearing in the expression (19). Since the piecewise cubic polynomials are discontinuous in the second derivative at the element attachments, the term $mv^2y''\delta(x-vt)$, when multiplied by the shape function $\boldsymbol{a}(x)$ and integrated over the element length, will produce functions of time which are discontinuous whenever the moving mass arrives at any point of attachment. Clearly, these discontinuities have nothing to do with the physics of the problem and are certain to invite trouble when one attempts to numerically integrate the time dependent equations via established algorithms. It is possible, of course, to use shape functions of higher degree to discretize the space variable, thus eliminating the discontinuities at the onset; but this is hardly consistent with the finite element method which should permit the use of even linear shape functions if need be. One is tempted to somehow smooth these discontinuities, yet this should not be done in a purely arbitrary fashion. Integrating the effects of these forces throughout the time domain through the use of Hamilton's law of varying action provides a consistent way to handle this problem.

While it is possible to handle the space and time finite element discretizations in one operation, the amount of computation and computer programming tend to become inordinately large. Moreover, there exist any number of finite element codes (e.g., NASTRAN) which can quickly accomplish much of the space discretization. It seems more efficient, therefore, to apply the finite element method in two steps, by first discretizing the space variable and then applying Hamilton's law to the resulting system of ordinary differential equations in time. For the case at hand, the differential equations governing the motion of the ith beam element turn out to be:

$$(p+\hat{m}c_1)\ddot{u}+\hat{m}c_2\dot{u}+(q+\hat{m}c_3)u+\hat{m}ga(vt)=0 \quad (20)$$

p and q are proportional to the usual mass and stiffness matrices for beam elements and have been evaluated many times in the literature. Here all of the beam elements are of the same length ℓ, and the displacement within the ith element is interpolated from $\boldsymbol{u}^i(t)$, a vector of end point displacements and velocities, i.e.,

$$y(x,t)=\boldsymbol{a}^T(\xi^i)\boldsymbol{u}^i(t) \qquad 0\leq\xi^i\leq 1 \quad (21)$$

where $\xi^i(x)=x/\ell-(i-1)$, a nondimensional element coordinate.

The c matrices in Eqs. (20) correspond to transverse, Coriolis, and centrifugal accelerations, respectively, and are defined for the ith element as follows:

$$c_1=\boldsymbol{a}(\xi^i)\boldsymbol{a}^T(\xi^i)\,|_{x=vt}$$

$$c_2=2v\boldsymbol{a}(\xi^i)\boldsymbol{a}'^T(\xi^i)\,|_{x=vt}$$

$$c_3=v^2\boldsymbol{a}(\xi^i)\boldsymbol{a}''^T(\xi^i)\,|_{x=vt} \quad (22)$$

It is noted that c_3 will be discontinuous at $\xi^i=0$ and $\xi^i=1$. The function $\hat{m}$ takes on the value of m only when the concentrated mass lies within the ith element, otherwise $\hat{m}$ is zero.

The element Eqs. (20) are combined in the usual way to form N equations of motion for the combined structure. Symbolically:

$$\boldsymbol{M}(t)\ddot{\boldsymbol{U}}+\boldsymbol{C}(t)\dot{\boldsymbol{U}}+\boldsymbol{K}(t)\boldsymbol{U}=\boldsymbol{F}(t) \quad (23)$$

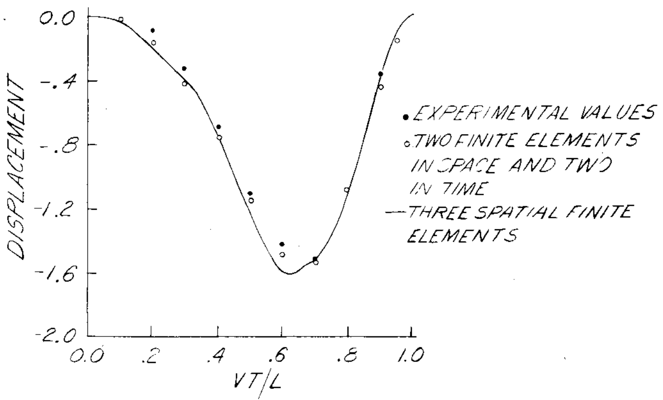

Fig. 2 Displacement of beam at location of moving mass.

Each of the matrices in Eq. (23) can be viewed as a conventional matrix of constant coefficients plus a time variant set of components which are active in a band along its main diagonal as the moving mass traverses the beam in time. For this system of equations Hamilton's law of varying action can be written:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left\{\int_{t_1}^{t_2}\{\delta\dot{U}_iM_{ij}\dot{U}_j+\delta U_i[(\dot{M}_{ij}-C_{ij})\dot{U}_j-K_{ij}U_j+F_i]\}\mathrm{d}t-\delta U_iM_{ij}\dot{U}_j\Big|_{t_1}^{t_2}\right\}=0 \tag{24}$$

It is interesting to observe the accuracy of solution which can be obtained from Eq. (24) using only two finite elements in space and two in time. A formulation using two elements in space results in a system of $N=4$ ordinary differential equations in time once the geometric support constraints have been applied. A two element formulation of these four equations for the time domain, followed by the application of all initial constraints in the manner summarized in Sec. IV, gives a final system of sixteen linear algebraic equations for solution. Figure 2 compares this solution with the experimental results of Ayre et al.[22] and a conventional finite element solution using three elements in the space domain followed by a time-integration of Eqs. (23) by Hamming's predictor-corrector algorithm.[23] The mass velocity in this case is $v=v^*/2$, $-v^*$ being the lowest velocity to cause resonance when the load is a moving weight only and the magnitude assigned to the moving mass is 25% of the total mass of the beam. (Other parametric values are the same as those in Ref. 22.) The displacements have been normalized with respect to the maximum deflection produced if the weight were applied statically at midspan, and L is the total beam length. In particular, one notes that the conventional solution obtained via three finite elements in space only, produces nonphysical discontinuities in the slope of the solution curve at $vt/L=1/3$, $2/3$. [The continuous data for generating this curve are obtained by interpolating the solution to Eq. (23) using Eq. (21).] No discontinuities of this sort can arise when finite elements in space and time are employed. Improved agreement with the experimental results is also observed.

Acknowledgments

The author is especially grateful for the interest and assistance of Mr. Royce Soanes of the Benet Computer Science Laboratory. All mathematics herein were performed using the MACSYMA (Project MAC's SYmoblic MAnipulation) system developed by the Mathlab Group of the M.I.T. Laboratory for Computer Science.

References

[1] Finlayson, B. A. and Scriven, L. E., "On the Search for Variational Principles," *International Journal of Heat & Mass Transfer*, Vol. 10, 1967, pp. 799-821.

[2] Lanczos, C., *The Variational Principles of Mechanics*, 3rd Ed., University of Toronto Press, Toronto, 1966, pp. 70-72.

[3] Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, No. 2, 1976, pp. 235-247.

[4] Zienkiewicz, O. C., *The Finite Element Method*, 3rd Ed., McGraw-Hill, New York, 1977, pp. 569-570.

[5] Kantorovich, L. V. and Krylor, V. I., *Approximate Methods of Higher Analysis*, 3rd Ed., Interscience Publishers, Inc., The Netherlands, 1964, pp. 258-303.

[6] Conte, S. D. and de Boor, C., *Elementary Numerical Analysis: An Algorithmic Approach*, 2nd Ed., McGraw Hill, New York, 1972, pp. 231-233.

[7] Hughes, T.J.R. and Liu, W. K., "Implicit-Explicit Finite Elements in Transient Analysis: Stability Theory," *Journal of Applied Mechanics*, Vol. 45, 1978, pp. 371-374.

[8] Belytschko, T. and Mullen, R., "Mesh Partitions of Explicit-Implicit Time Integration," *Formulations and Computational Algorithms in Finite Element Analysis*, edited by K. J. Bathe, J. T. Oden, and W. Wunderlich, M.I.T. Press, Cambridge, Mass., 1977.

[9] Serbin, S., personal communication, University of Tennessee, March 1981.

[10] Argyris, J. H. and Scharpf, D. W., "Finite Elements in Time and Space," *Nuclear Engineering and Design*, Vol. 10, 1969, pp. 456-464.

[11] Fried, I., "Finite-Element Analysis of Time-Dependent Phenomena," *AIAA Journal*, Vol. 7, June 1969, pp. 1170-1172.

[12] Riff, R., Weiler, T., and Baruch, M., "Spare-Time Finite Elements for Structural Dynamic Analysis," Technion, Israel Institute of Technology, Dept. of Aerospace Engineering, TAE Rept. No. 345, Nov. 1978.

[13] Riff, R. and Baruch, M., "Hamilton's Principle, Hamilton's Law, 6 Correction Formulations," Technion, Israel Institute of Technology, Dept. of Aerospace Engineering, TAE Rept. No. 403, March 1980.

[14] Mierovitch, L., *Methods of Analytical Dynamics*, 1st Ed., McGraw-Hill, New York, 1970, p. 65.

[15] Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 13, Sept. 1975, pp. 1154-1157.

[16] Hamilton, W. R., "Second Essay on a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London*, 1835, pp. 95-144.

[17] Courant, R., "Variational Methods for the Solution of Problems of Equilibrium and Vibrations," *Bulletin of American Mathematical Society*, Vol. 49, Jan. 1943, pp. 1-23.

[18] Tiersten, H. F., "Natural Boundary and Initial Conditions from a Modification of Hamilton's Principle," *Journal of Mathematical Physics*, Vol. 9, Sept. 1968, pp. 1445-1450.

[19] Gurtin, M. E., "Variational Principles for Linear Elastodynamics," *Archive for Rational Mechanics and Analysis*, Vol. 16, 1964, pp. 34-50.

[20] Argyris, J. H. and Scharpf, D. W., "Finite Elements in Time and Space," *Nuclear Engineering and Design*, Vol. 10, 1969, p. 459.

[21] Simkins, T. E., "Unconstrained Variational Statements for Initial and Boundary-Value Problems," *AIAA Journal*, Vol. 16, June 1978, pp. 559-563.

[22] Ayre, R. S., Jacobsen, L. S., and Hsu, C. S., "Transverse Vibration of One and Two Space Beams Under the Action of a Moving Mass Load," *Proceedings of First National Congress on Applied Mechanics*, ASME, June 1951, pp. 81-90.

[23] Ralston, A. and Wilf, H. S., *Mathematical Methods for Digital Computers*, Wiley and Sons, N.Y., London, 1960, pp. 95-109.